

# MATH 8500 Algorithmic Graph Theory, Spring 2017, OSU

## Lecture 9: Treewidth

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Many problems in graph theory are NP hard. However this sometimes can be “bypassed” for some special kinds of graph. In previous lectures, we saw how *Szemerédi’s regularity lemma* can be used to solve (or at least approximate a solution of) some NP-problems on dense graphs. Similarly, other kinds of problems can be solved in polynomial time over trees, as we saw with the Traveling Salesman Problem.

This is the case with many more problems, were having a graph that “looks like” a tree allows to solve them in a more efficient way through dynamic programming: solving smaller problems first and combine them to get a global solution. The **treewidth** of a graph somehow measures the likeness of said graph to a tree, and usually having a small treewidth means faster algorithms.

### Definitions

- A **tree decomposition** of a graph  $G$  is a pair  $(T, X)$ , where  $X$  is a family of subsets of  $V(G)$  and  $T$  is a tree with  $V(T) = X$ , such that:

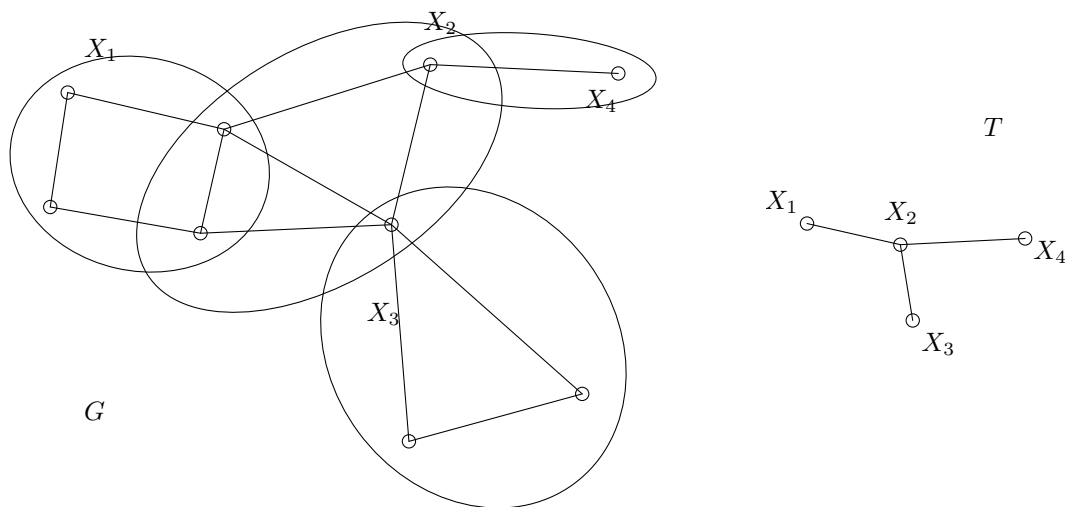
1.  $\bigcup_{X_i \in X} X_i = V(G)$
2.  $\forall (u, v) \in E(G), \exists X_i \in X$  such that  $\{u, v\} \subset X_i$ .
3. If  $X_i, X_j, X_k \in V(T)$  are such that  $X_k$  is in the path from  $X_i$  to  $X_j$  in  $T$ , then  $X_i \cap X_j \subseteq X_k$ .

- The  $X_i$ ’s above are referred to as **bubbles**.
- The **width** of a tree decomposition  $(X, T)$  is  $\max_{X_i \in X} \{|X_i| - 1\}$ .
- The **treewidth** of  $G$ , denoted  $\text{tw}(G)$  is the minimum width of any tree decomposition of  $G$ .

**Remark:** Condition 3) in the definition of tree decomposition is equivalent to:

$$\forall v \in V(G), \text{ the set of bubbles that contain } v \text{ form a connected subtree of } T$$

### Example



That  $T$  satisfies conditions 1 and 2 is clear. Also,  $T$  satisfies condition 3 trivially, since the intersection of two bubbles is empty unless they are neighbors.

## Theorem

Let  $G$  be a connected graph with at least one edge. Then,  $\text{tw}(G) = 1$  if and only if  $G$  is a tree.

*Proof.*

( $\Leftarrow$ ):

Let  $G$  be a connected tree with at least one edge. Since  $G$  contains an edge, its two vertices must be in the same bubble for any tree decomposition, so  $\text{tw}(G) \geq 1$ . Consider the tree decomposition given by

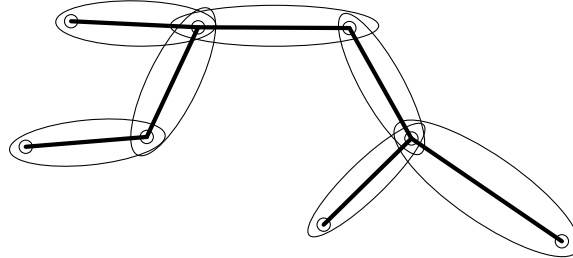
- $X = E(G) \cup \{\{v\} : v \in V(G)\}$
- $E(T) = \{\{e, \{v\}\} : e \in E(G), v \in V(G)\}$

By construction of  $(T, X)$ ,  $V(G) = \bigcup X$  and  $\forall e = \{u, v\} \in E(G), \{u, v\} \in e$ , with  $e \in X$ . So it satisfies properties 1) and 2) of the definition. For property 3), note that for every vertex  $v \in G$ , it is contained in the bubbles  $\{v\}$  and some  $e$ 's to which it is incident, so the subtree of  $T$  is connected. Hence  $(X, T)$  is a tree decomposition with width 1. Therefore  $\text{tw}(G) = 1$ .

( $\Rightarrow$ ):

Let  $G$  be a connected graph with  $\text{tw}(G) = 1$ . Let  $(T, X)$  be a tree decomposition of  $G$  with width 1, so  $|X_i| \leq 2 \forall i$ . It is possible to build another tree decomposition  $(T', X')$  with all bubbles of size exactly 2, by combining any vertex singleton with a connected bubble of an incident vertex, which must exist because of conditions 2 and 3.

Since  $T'$  is a tree, it is also possible to prove that all bubbles must be edges of  $G$ , otherwise, since  $G$  is connected and because of condition 3,  $T'$  should have a cycle. Finally, because of condition 3, two bubbles subtend an edge in  $T'$  if and only if they have a common vertex (as edges of  $G$ ). Therefore  $T'$  must be similar to the next diagram, and so  $G$  must be a tree.



□

**Alternative Proof:** The proof of the ( $\Rightarrow$ ) direction above, outlines a transformation for the tree decomposition of  $G$  into another that makes it clear that  $G$  itself must also be a tree. While correct, I find that filling in the details of the given outline is not trivial. The following proof by contradiction is easier to follow.

Let  $G$  be a connected graph with  $\text{tw}(G) = 1$ . Suppose by way of contradiction that  $G$  is not a tree, so it has a cycle  $v_1v_2 \cdots v_nv_1$ , with  $n \geq 1$ . Let  $(T, X)$  be a tree decomposition of  $G$  of width 1, so each bubble has at most 2 vertices. By condition 2, for every edge  $\{v_1, v_2\}, \{v_2, v_3\} \cdots \{v_{n-1}, v_n\}, \{v_n, v_1\}$  there must be a bubble in  $X$  containing it.

Hence, there exist bubbles  $X_1, \cdots, X_n \in X$  such that  $X_1 = \{v_1, v_2\}, \cdots, X_n = \{v_n, v_1\}$ . Since  $T$  is a tree, there is a path between  $X_1$  and  $X_2$ , and since  $v_2 \in X_1 \cap X_2$ , by condition 3, all bubbles in such path must have  $v_2$ . Similarly, there must be a path of bubbles containing  $v_3$  between  $X_2$  and  $X_3$ . Note

all the bubbles between  $X_2$  and  $X_3$  must be different to the ones between  $X_1$  and  $X_3$ , otherwise  $T$  would have a cycle. Thus there is a path of bubbles between  $X_1$  and  $X_3$ .

Continuing with the same process, there is a path between  $X_1$  and  $X_n$ . However,  $X_2 = \{v_2, v_3\}$  is part of this path, and  $v_1 \notin X_2$ , but by condition 3, there must be a path between  $X_1$  and  $X_n$  of bubbles containing  $v_1$ . But this new path either intersects the original path somewhere different to the endpoints, forming a cycle, or completes a cycle in  $X_1$ . Thus,  $T$  is not a tree, which is a contradiction.  $\square$

### Claim

For any cycle  $G$ ,  $\text{tw}(G) = 2$

*Proof.*

It is enough to construct a tree decomposition of width 2. Suppose  $V(G) = \{1, 2, \dots, n\}$  and  $E(G) = \{\{1, 2\}, \{2, 3\}, \dots, \{n, 1\}\}$ . Then let

$$X = \{X_1 = \{1, 2, 3\}, X_2 = \{1, 3, 4\}, X_3 = \{1, 4, 5\}, \dots, X_{n-2} = \{1, n-1, n\}\}$$

$$E(T) = \{\{X_1, X_2\}, \{X_2, X_3\}, \dots, \{X_{n-3}, X_{n-2}\}\}$$

Clearly  $(T, X)$  is a valid tree decomposition of width 2, so  $\text{tw}(G) = 2$ .  $\square$

### Claim

For the complete graph on  $n$  vertices,  $K_n$ , we have  $\text{tw}(K_n) = n - 1$ .

*Proof.*

$\text{tw}(G) \leq n - 1$  trivially, by considering only one bubble. To prove the converse inequality recall the following property:

#### Helly-type property of trees

Let  $T_1, \dots, T_k$  be a collection of subtrees of  $T$  such that  $T_i \cap T_j \neq \emptyset$  whenever  $i \neq j$ . Then  $\bigcap_i T_i \neq \emptyset$ .

Let  $(T, X)$  be a tree decomposition of  $G$ . Suppose  $V(K) = \{1, \dots, n\}$ . For every  $i \in V(K)$ , let  $T_i$  be the subtree of  $T$  formed by the bubbles that contain  $i$ . Then, for any  $j \neq i$ , since  $K$  is complete,  $\{i, j\} \in E(K)$ , so there exists a bubble  $X_{ij} \ni \{i, j\}$ , and  $X_{ij} \in T_i \cap T_j \neq \emptyset$ . Thus, by the Helly property, there exists a bubble  $X' \in \bigcap_i T_i$ , so  $i \in X' \forall i \in V(K)$ . That is,  $X' = V(K)$ , hence  $\text{tw}(G) \geq n - 1$ . Therefore,  $\text{tw}(G) = n - 1$ .  $\square$

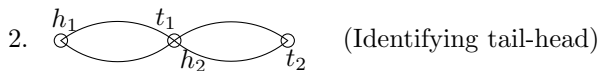
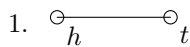
The next result shows a characterization of all graphs of treewidth 2.

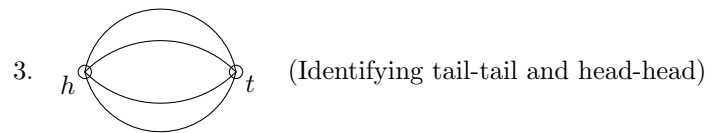
### Theorem

A graph has treewidth 2 if and only if every biconnected component is series-parallel.

### Definition

- **Biconnected component** Maximal subgraph such that there is *no* cut-vertex, i. e. there doesn't exist a vertex  $v$  such that  $G \setminus \{v\}$  is not connected.
- A **series parallel graph** with **head**  $h$  and **tail**  $t$  is one of the following





Or constructed inductively from the cases above.