MATH 8500 Algorithmic Graph Theory, Spring 2017, OSU
Lecture 7: Maximum Matching
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1. Maximum Matching in General Graphs

Definition 1. *A matching (M) is a set of independent edges in a graph without common vertexes.*

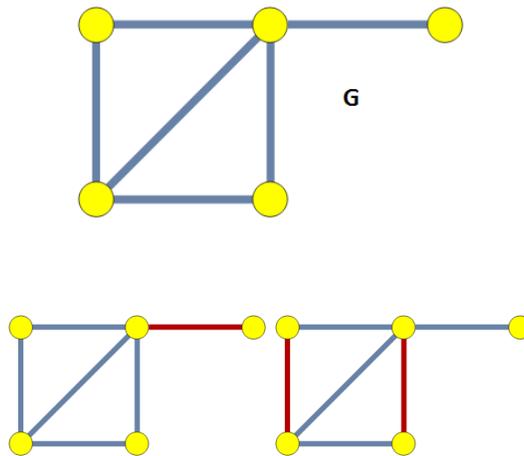Some examples of matchings are shown in the following figure:

Figure 1. Graph G and two types of matching of G.

2. Maximum Matching Problem

In the Maximum Matching problem the goal is to find a matching of maximum cardinality in a given graph.

MAXIMUM MATCHING PROBLEM

Input: An undirected and unweighed graph $G = (V, E)$.
Goal: A matching M in graph G which maximizes $|M|$.

We talked about maximum matching in bipartite graphs which is a special case of matching in the previous section, now we want to illustrate an algorithm for finding maximum matching in general graphs, which is more complicated than finding the maximum matching in the bipartite graphs.

Definition 2. *Augmenting Path*

In a graph a path is called an augmenting path if it starts with an exposed vertex and then ends to an exposed vertex. Edges in the path alternatively belong to matched and unmatched edges.

The following theorem which we proved in the previous section for bipartite graph still holds true for general graphs:

Theorem 1. *A maximum matching M in a graph G is maximum if and only if there is no augmenting path in G.*

1. The path starts from an exposed vertex and reaches to another exposed vertex (Figure 2).
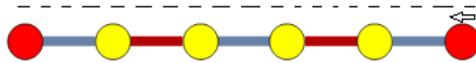
Figure 2. The path starts from an exposed vertex (red vertex) and reaches to another exposed vertex (red vertex)

2. The path starts from an exposed vertex and at the end of traversal comes back to the starting exposed vertex, then this path is not an augmenting path because there is only one exposed vertex. The following figure (Figure 3) illustrates it better.
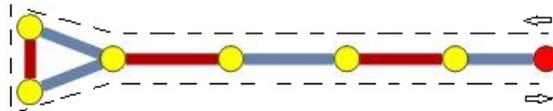
Figure 3. The path starts from an exposed vertex (red vertex) back to the previous exposed vertex.

Definition 3. *Flower*

In a graph, a structure that consists of a path, an exposed vertex as starting point and an even number of edges, ending to a cycle of odd length, where the path and the cycle separately have the maximum number of matching edges, is called a Flower. The cycle is called Blossom and the path is called Stem. (see Figure 4)
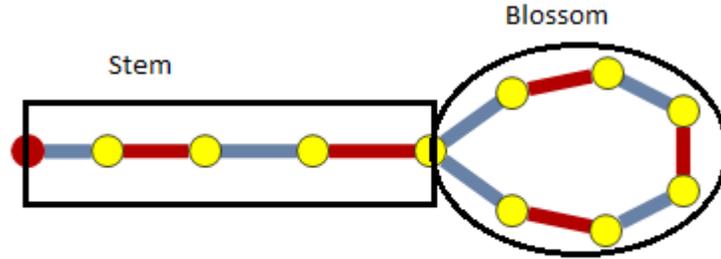
Figure 4. Flower, Blossom and Stem.

Now by defining that structure we can define the following Algorithm:

ALGORITHM: (Maximum Matching in General Graphs)

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1. $M = \phi$
2. Repeat until no paths and no flowers exists.
3.   If there exists an augmenting path P (w.r.t M) then let $M = M \triangle P$[1].
4.   If there exists a flower F with stem Q and blossom B, then contract B and let $M = M \triangle Q$.
5. end

Lemma 1. *M is a maximum matching in G if and only if $M \setminus B$ is a maximum matching in $G \setminus B$.*

Proof.

   $\Rightarrow$: **If M is a maximum matching in G then $M \setminus B$ is a maximum matching in $G \setminus B$.**

   Proof by contradiction:

   We assume that there exists a maximum matching N in $G \setminus B$, where $|N| > |M \setminus B|$. Thus there exists a matching N' in G with the size of $|N'| = |N| + \lfloor \frac{|B|}{2} \rfloor$. Therefore, we can obtain

$$|N'| = |N| + \lfloor \frac{|B|}{2} \rfloor > |M \setminus B| + \lfloor \frac{|B|}{2} \rfloor = |M|,$$

which contradicts the assumption that $|M|$ is a maximum matching for G.

   $\Leftarrow$: **If $M \setminus B$ is a maximum matching in $G \setminus B$ then M is a maximum matching in G.**

   Proof by contradiction:

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   [1]$M \triangle P := (M \setminus P) \cup (P \setminus M).$

If M is not a maximum matching in G then there exists an augmenting path P (w.r.t. M). Suppose that endpoints of P are u and v.

Two things may happen:

1. $P \cap B = \phi$, then P is an augmenting path w.r.t $M \backslash B$ which is a contradiction with the assumption that M is a maximum matching in G.

2. $P \cap B \neq \phi$, therefore B has at most one exposed vertex. We assume w.l.o.g that $u \notin B$. Let w be the first vertex in $P \cap B$. Let Q be the sub-path of P from u to w. Let Q be the vertex into which B is connected. Now we claim that b is an exposed vertex in $G \setminus B$. Thus Q is an augmenting path in $G \setminus B$. Thus $M \setminus B$ in not a maximum matching in $G \setminus B$, which is a contradiction with the assumption. $\square$

3. FINDING AUGMENTING PATHS OR FLOWERS IN A GRAPH

We start from exposed vertexes and sign them as even, and their children as odd (even and odd numbers indicate whether the distance between those vertexes from the related exposed vertex is an odd or even number), and repeat. Two things will happen:

1. If the vertex is a signed exposed vertex in another tree, then it means that an augmenting path exists. (Figure 5)
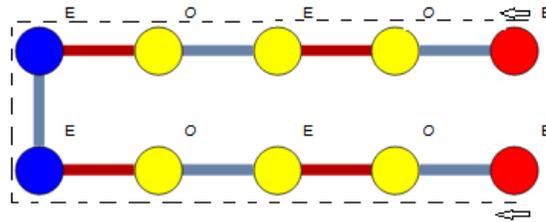

Figure 5. An augmenting Path.

2. If the vertex is a signed even vertex in the same tree, then it means that it is a flower. (Figure 6)
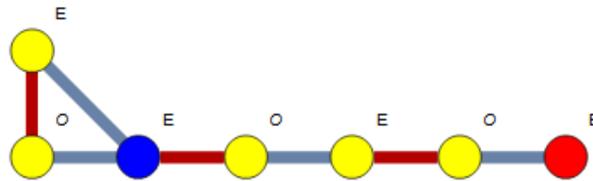

Figure 6. A flower.