MATH 8500 Algorithmic Graph Theory, Spring 2017, OSU
Lecture 5: ℓ-Way Cut Continued
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1 ℓ-Way Cut Continued

Recall from previous lectures the following problem statement

> **ℓ-Way Cut:**
> **Input:** $G = (V, E)$, $n = |V|$
> **Goal:** Find a partition $S_1, \ldots, S_\ell$ of V maximizing $|E(S)|$, where
>
> $$E(S) = \{\{u, v\} \in E : u \in S_i \text{ and } v \in S_j \text{ for } i \neq j\}$$

and the following definition

> **Definition 1.1.** *We say that a partition $V_1, \ldots, V_k$ of V is "ϵ-sufficient" if*
>
> $$|\Delta(S,T)| \leq \epsilon \cdot n^2, \forall S, T \subseteq V, S \cap T = \emptyset,$$
>
> *where*
>
> - $\Delta(S,T) = e(S,T) - \sum_{i=1}^{k} \sum_{j=1}^{k} d_{i,j} |S_i||T_i|$
> - $e(S,T) = |E(S,T)|$
> - $d_{i,j} = d(V_i, V_j) = e(V_i, V_j)/|V_i||V_j|$.

Consider the example of Figure 1. Here we have the following:

- $e(S,T) = 3$
- $d_{1,1} = d_{2,2} = 0$
- $d_{1,2} = d_{2,1} = 4^2/4^2 = 1$

Thus, in this case we have that

$$\sum_{i=1}^{k} \sum_{j=1}^{k} d_{i,j} |S_i||T_i| = d_{1,2}|S_1||T_2| + d_{2,1}|S_2||T_s| = 2 \cdot 1 + 1 \cdot 1 = 3$$

and therefore Figure 1 is 0-sufficient.

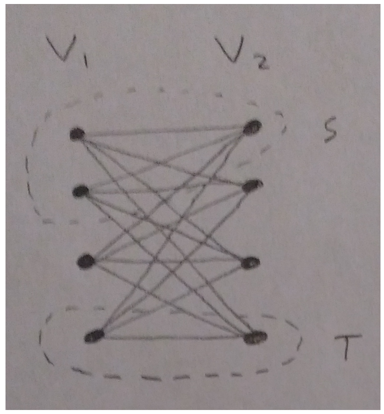

Figure 1: An example of a 0-sufficient partitioning.

Theorem 1.2. *There is a randomized polynomial time algorithm which given an n-vertex graph G, with probability at least $3/4$, computes a partition S_ϵ such that*

$$|E(S_\epsilon)| \geq |E(S^*)| - \epsilon n^2,$$

where S^* is an optimal partition.

Proof. Compute an ϵ-sufficient partition $V_1, \ldots, V_k$ of G using [Alon et al.]. Let $S = S_1, \ldots, S_\ell$ be an ℓ-way cut of G. Let $S_{i,r} = S_r \cap V_i$, and $T_{i,r} = V_i \setminus S_{i,r}$. We have that

$$2|E(S)| = \sum_{r=1}^{\ell} e(S_r, V \setminus S_r).$$

By the definition of ϵ-sufficient, we have that

$$2|E(S)| = \left(\sum_{r=1}^{\ell} \sum_{i=1}^{k} \sum_{j=1}^{k} d_{i,j} |S_{i,r}||T_{j,r}| \right) + \Theta$$

where $\Theta \leq \ell \epsilon n^2$. Let $v_j = |V_j|$, $\rho = \lfloor \frac{\epsilon n}{k} \rfloor$, $\bar{v}_j = \lfloor \frac{v_j}{\rho} \rfloor$, $n_{i,r} = |S_{i,r}|$, $\bar{n}_{i,r} = \lfloor \frac{n_{i,r}}{\rho} \rfloor$. We have

$$|n_{i,r}(v_j - n_{j,r}) - \rho^2 \bar{n}_{i,r}(\bar{v}_j - \bar{n}_{j,r})| \leq \rho(v_i + v_j).$$

Thus

$$\left| \sum_{r=1}^{\ell} \sum_{i=1}^{k} \sum_{j=1}^{k} d_{i,j} n_{i,r}(v_j - n_{j,r}) - \rho^2 \sum_{r=1}^{\ell} \sum_{i=1}^{k} \sum_{j=1}^{k} d_{i,j} \bar{n}_{i,r}(\bar{v}_j + \bar{n}_{j,r}) \right| \leq 2\epsilon \ell n^2.$$

Therefore,

$$\left| 2|E(S)| - \rho^2 \sum_{r=1}^{\ell} \sum_{i=1}^{k} \sum_{j=1}^{k} d_{i,j} \bar{n}_{i,r}(\bar{v}_j + \bar{n}_{j,r}) \right| \leq 3\epsilon \ell n^2.$$

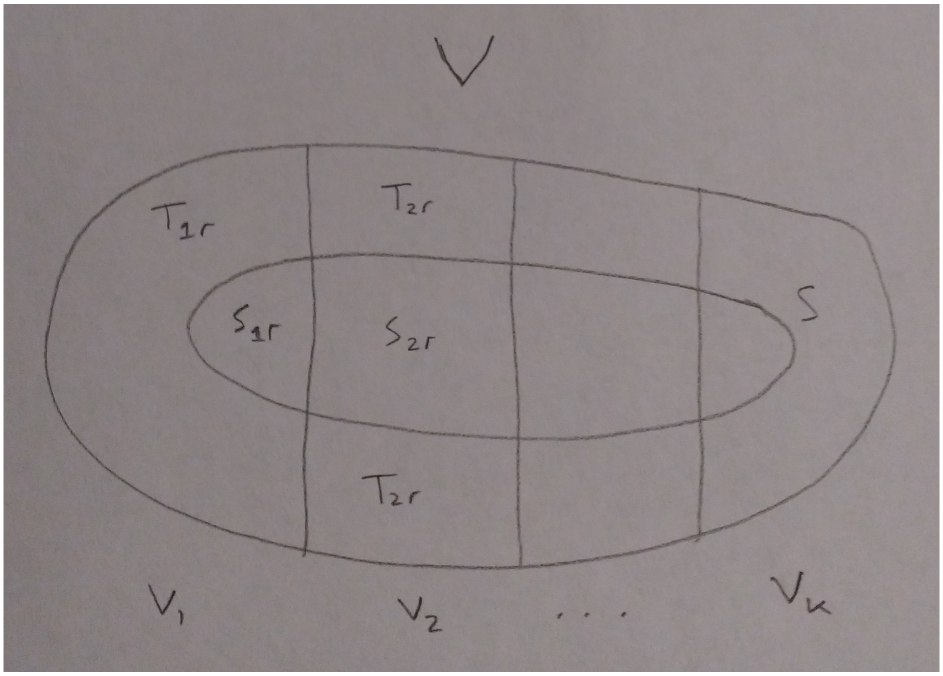

Figure 2: Illustration of $S_{i,r}$ and $T_{i,r}$. The outer circle corresponds to V, and the inner circle to S. The vertical lines mark the partitions $V_1, \dots, V_k$.

Each $\bar{n}_{i,r}$ has at most $\frac{1}{\epsilon}$ different possible values. There are $k\ell$ variables $n_{i,r}$. Therefore, there are $(\frac{1}{\epsilon})^{k\ell}$ possibilities. It is sufficient to iterate over all these possibilities, and select the instance maximizing

$$\sum_{r=1}^{\ell} \sum_{i=1}^{k} \sum_{j=1}^{k} d_{i,j} \bar{n}_{i,r} (\bar{v}_j + \bar{n}_{j,r}).$$

□

Recall that we have the following lemma from a previous lecture:

Lemma 1.3. *If* $|E| \geq \gamma n^2$ *then* $|E(S^*)| \geq (1 - \frac{1}{\ell})|E| \geq \gamma(1 - \frac{1}{\ell})n^2$.

Using this lemma and the theorem above, we have the following corollary:

Corollary 1.4. *For all* $\epsilon > 0$ *there exists a polynomial time algorithm that computes a* $(1 + \epsilon)$*-approximation* ℓ*-way cut on dense graphs.*

Thus, we have a **PTAS** (**P**olynomial **T**ime **A**pproximation **S**cheme) for ℓ-Way Cut on dense graphs.