5339 - Algorithms design under a geometric lens
Spring 2014, CSE, OSU
Lecture 4: Embeddings into ℓ_p space

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Bourgain's embedding

Theorem (Bourgain '85)

Any n-point metric space (X, ρ) *admits an embedding into* ℓ_2 *with distortion* $O(\log n)$.

A special case of Bourgain's theorem

Lemma

Any finite sub-metric of ℓ_2 admits an isometric embedding into ℓ_1.

Special case of Bourgain's theorem (this lecture):

Theorem

Any n-point metric space (X, ρ) admits an embedding into ℓ_1 with distortion $O(\log n)$.

The cut-cone

Let X be a set.
A metric (X, ρ) is a *cut pseudo-metric* if there exists a set $Y \subset X$, such that for any $x, y \in X$

$$\rho(x,y) = \begin{cases} 1 & \text{if } |Y \cap \{x,y\}| = 1 \\ 0 & \text{otherwise} \end{cases}$$

ℓ_1 and the cut-cone

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ℓ_1 and the cut-cone

Let $X \subset \mathbb{R}^d$, for some $d > 0$.
Then, there exists a collection $(X, \rho_1), \ldots, (X, \rho_k)$ of cut pseudo-metrics, and positive reals $\alpha_1, \ldots, \alpha_k$, such that

$$\sum_{i=1}^{k} \alpha_i = 1,$$

and for any $x, y \in X$

$$\|x - y\|_1 = \sum_{i=1}^{k} \alpha_i \cdot \rho_i(x, y)$$

Corollary

Every finite ℓ_1 metric can be represented as a convex combination of cut pseudo-metrics.

ℓ_1 and the cut-cone

Proof sketch.

It suffices to prove the assertion for $d = 1$, and apply the argument on every dimension independently.

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Let $X = \{x_1, \ldots, x_n\}$, where

$$x_1 \leq x_2 \leq \ldots \leq x_n.$$

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Let $X_i = \{x_1, \ldots, x_i\}$.
Let ρ_i be the cut-pseudometric induced by X_i, i.e. for any $j < r$, $\rho_i(x_j, x_r) = 1$ if and only if $j \leq i$, and $r \geq i + 1$.

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Let $\alpha_i = x_{i+1} - x_i$.
Then, for any $x, y \in X$, $\rho(x, y) = \sum_{i=1}^{n-1} \alpha_i \cdot \rho_i(x, y)$. □

ℓ_1 and the cut cone

Conversely:

Lemma

Every convex combination of cut pseudo-metrics, admits an isometric embedding into ℓ_1 (i.e. with distortion 1).

Embedding trees into ℓ_1

Lemma

The shortest path metric of any tree T admits an isometric embedding into ℓ_1 (i.e. with distortion 1).

Proof.

Proof sketch For every edge e of T, let X_e, Y_e be the vertices of the two connected components of $T \setminus e$.

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Observe that for any $x, y \in V(T)$, we have

$$d_G(x, y) = \sum_{e \in E(T)} \rho_e(x, y) \cdot \text{length}(e)$$

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Note that any cut pseudo-metric admits an isometric embedding into $\mathbb{R}^1$.
Thus, any convex combination of cut pseudo-metrics admits an isometric embedding into ℓ_1.
Thus, any tree metric admits an isometric embedding into ℓ_1. □

From random trees to ℓ_1

Theorem (Fakcharoenphol, Rao, Talwar '04)

Any n-point metric admits a random embedding into a distribution over trees, with distortion $O(\log n)$

In other words:

Corollary

Any n-points metric admits an embedding into a convex combination of tree metrics, with distortion $O(\log n)$.

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Any n-point metric admits an embedding into ℓ_1 with distortion $O(\log n)$.

Proof.

Embed the metric into a convex combination of trees.
Embed each tree into ℓ_1.
Concatenate the embeddings, weighted/scaled by the probability of the corresponding tree. □